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Excessive distribution of quantum entanglement

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We classify entanglement distribution protocols based on whether or not entanglement gain is observed with respect to communicated and initial entanglement. We call a protocol nonexcessive if the gain of entanglement is bounded by the communicated entanglement and excessive if it violates this bound. We present examples of excessive protocols that achieve significant gain, independently of the presence of the initial and (or) communicated entanglement. We show that, for certain entanglement measures, excessive entanglement distribution is possible even with pure states, which sheds light on the possibility of formulating a unifying approach to quantifiers of quantum correlations. We point out a "catalytic" effect, where a protocol is turned into an excessive one by sending an intermediate particle (which does not change the initial entanglement) in advance of the designated carrier. Finally, we analyze the protocols in noisy scenarios and show that, under suitable conditions, excessive distribution may be the only way to achieve entanglement gain.

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I. INTRODUCTION

Quantum entanglement is not only an essential concept of quantum mechanics but also "a new resource as real as energy" [1]. Distributing entanglement between two distant laboratories is crucial for quantum information processing as exemplified by cryptography [2], dense coding [3], and teleportation [4]. Nonetheless, limits on entanglement distribution have only recently been studied and are not fully understood.

Remarkably, communication of entanglement is not necessary to create an entangled network of local nodes [5]. This finding has attracted considerable attention, resulting in several theoretical proposals [6–9] and inspiring test-bed experimental realizations [10–13]. On the other hand, it has generated great curiosity about what really limits the distribution, if not the carried entanglement. In Refs. [14] and [15] it has been shown that quantum discord is a necessary condition for a successful distribution, providing an upper bound to the amount of entanglement generated. However, the presence of discord in the carrier is not a sufficient condition, and, e.g., Ref. [16] investigates other limitations of the resources, highlighting a link to the dimensionality and the rank of the state of the carrier system for distribution with separable states. A recent work [17] further investigates the role of carried entanglement and other quantum correlations in the presence of noise. In this case, it is shown that the optimal strategy may depend on the entanglement measure at hand.

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Our work aims to contribute along similar directions by suggesting a systematic characterisation of entanglement-distribution protocols and illustrating explicit examples where such schemes could be useful. We present two classes of distribution protocols: in *direct* distribution schemes, entanglement is first established *locally* in one laboratory by direct interaction of two subsystems and then distributed by sending one of them to a remote node of a network. In *indirect* schemes, one starts with the subsystems already apart and uses ancillary systems as communication channels between the laboratories to establish entanglement between them. This class encompasses, among others, the intriguing protocols that rely only on separable ancillary carriers mentioned above [5–13]. They reveal that, even if no entanglement is being communicated, i.e., ancillary systems are at all times separable from the core systems, entanglement in the state of the latter can grow. The existence of such protocols is our main motivation to further subdivide the class of indirect protocols. We call a protocol *excessive* if the entanglement gain exceeds the amount of communicated entanglement and *nonexcessive* otherwise. Note that the initial and the communicated entanglement can both be present in our scenario.

We now provide a concise summary of the key results achieved in our work.

(1) We show that excessive entanglement gains are possible with pure states (in certain dimensions) when entanglement is measured by negativity or logarithmic negativity. Therefore there is no natural discord-like quantity on equal footing with negativity. This means that a discord-like quantity cannot reduce to negativity for pure states and bound entanglement gain in general protocols.

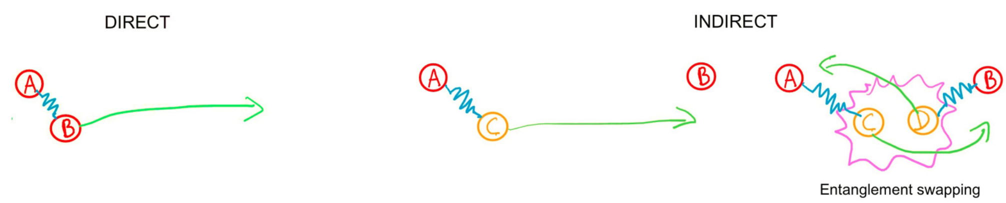

FIG. 1. Direct and indirect protocols for entanglement distribution. In the direct protocol, the systems of interest get entangled via mutual interaction. In the indirect protocol they get entangled via interactions with ancillary systems. As examples, we draw a unidirectional protocol with ancilla traveling from one laboratory to the other and an entanglement swapping scheme with Bell measurement conducted on the ancillae.

(2) We present analytical examples of excessive protocols for all possible combinations of zero and nonzero initial and communicated entanglement.

(3) We show that, under suitable conditions, transmitting a particle to a remote receiver before the designated carrier, an operation that does not change the degree of initial entanglement, can make the protocol excessive and significantly improve the degree of entanglement gain. We call this phenomenon *catalysis of excessiveness*.

(4) As entanglement can be gained without communicating it, one might ask whether excessive protocols allow for entanglement gain via entanglement-breaking (EB) channels. In principle, this would provide an operational meaning to our classification because only excessive protocols (with a separable carrier) could achieve entanglement gain. However, the answer to this question is negative and we demonstrate that entanglement gain is always impossible with EB channels.

(5) We complement the analytical results illustrated above with numerical studies showing that excessive protocols emerge naturally in the presence of weak channel noise and allow for greater robustness. Quite remarkably, we demonstrate that they are often the only way of increasing entanglement.

The results summarized above and presented in greater detail in the following sections suggest that excessive protocols have considerable purpose and a vast range of applicability, often advantageous with respect to entanglement distribution via separable states. They thus deserve to be considered a distinct class of protocols with a distinctive role in the panorama of quantum communication and entanglement distribution.

In what follows we provide examples of protocols that belong to both excessive and nonexcessive classes. Our study addresses both pure and mixed states, under both ideal and noisy conditions. In Sec. II we describe in detail the classifications above. Section III demonstrates that, for certain measures of entanglement, excessive distribution is possible even with pure states. We also provide cases of excessive protocols in which considerable entanglement gain is achieved via both separable and entangled carriers. In Sec. IV we investigate a possible advantage of indirect and excessive protocols in noisy environments.

II. ENTANGLEMENT DISTRIBUTION PROTOCOLS

We start by describing in detail the two classes of entanglement distribution protocols addressed in this work and give examples of typical members of each class.

A. Direct and indirect protocols

Direct protocols for distributing entanglement are the most straightforward ways of increasing entanglement between distant laboratories: all they entail is the preparation of entangled states in one laboratory and the transmission of one subsystem to a distant laboratory. In an indirect protocol, on the other hand, one requires the use of additional systems to entangle the main ones, which are typically already located in distant laboratories. The difference between direct and indirect protocols is illustrated in Fig. 1.

The simplest example of an indirect entanglement distribution protocol is first to entangle an ancilla with a particle present in one of the remote laboratories and then to transmit it to the other laboratory. The protocol would be completed by swapping the state of the ancilla and that of the local particle at the remote laboratory. A well-known example of this kind of protocol is entanglement swapping [18], where entanglement initially present in the state of two system-ancilla pairs is teleported to the systems alone [19,20].

Under small experimental imperfections and low-noise channels, the implementation of indirect distribution protocols is likely too demanding to be practically useful. However, for imperfect operations and situations where noise cannot be ignored, it is quite natural to conceive that an indirect protocol might be more advantageous than direct distribution schemes. Such intuition is reinforced by the results presented in this paper.

We focus on simple indirect protocols with ancillae transmitted in one way, i.e., from one laboratory to the other. We calculate the final interlaboratory entanglement achieved through the implementation of a given protocol, thus including the core system and the ancillae, instead of focusing on the entanglement between the core subsystems only. Although, in principle, these are two different quantities, it has been proven in Ref. [15] that entanglement can be localized into the state of the core system as long as certain dimensionality conditions are satisfied, which hold for most of the cases discussed in this paper.

B. Excessive and nonexcessive protocols

Our second classification divides the protocols with respect to the amount of entanglement gained compared to the communicated entanglement. Figure 2 presents quantities relevant to this classification. We consider the change of entanglement between the laboratory of Alice and that of Bob caused solely by the exchange of an ancillary carrier system between them (particle C in Fig. 2). We define *communicated entanglement*

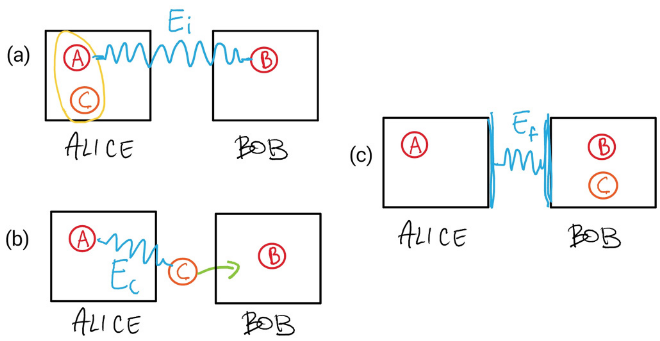

FIG. 2. Scenario of excessive and nonexcessive entanglement distribution protocols. We study entanglement gain between the laboratories caused solely by the communication of particle C from Alice to Bob. (a) Initially entanglement between the laboratories is given by $E_{AC:B}$. (b) The communicated entanglement is taken to be the entanglement between the carrier and the other subsystems, i.e., $E_{AB:C}$. (c) Finally, the entanglement between the laboratories is given by $E_{A:CB}$. The protocol is excessive if $E_{A:CB} - E_{AC:B} > E_{AB:C}$, meaning that the gain is greater than what was communicated. Otherwise we call it nonexcessive.

E_{com} as the entanglement between C and the systems at the laboratories, which are dubbed A and B. Therefore, we take $E_{\text{com}} = E_{AB:C}$. The change of entanglement resulting from this communication step is intended as the difference between the interlaboratory entanglement when C is with Bob and that when C is with Alice, i.e.,

$$\Delta E \equiv E_{A:CB} - E_{AC:B}. \tag{1}$$

A protocol is called excessive or nonexcessive depending on how ΔE compares with E_{com}. In particular, we have

$$\begin{aligned} \Delta E &\leqslant E_{\text{com}}, \quad \text{nonexcessive protocol;} \\ \Delta E &> E_{\text{com}}, \quad \text{excessive protocol.} \end{aligned}$$

Therefore, in an excessive protocol the entanglement gain exceeds the limit set by the communicated entanglement. As we show, this property is dependent on the choice of entanglement monotone.

The examples of indirect protocols given above are all nonexcessive. Other notable classes of nonexcessive indirect protocols are presented later. The existence of excessive protocols was first pointed out in Ref. [5] and further examples are presented in Refs. [6–9]. However, to the best of our knowledge, an analytical example of an excessive protocol with nonzero communicated entanglement is presented here for the first time.

In addition to the entanglement change between laboratories, it is interesting to investigate the change of entanglement in the principal system. In Fig. 2, the latter is composed of particles A and B, which are stationary in the laboratories of Alice and Bob. As initial entanglement in the principal system E_i we naturally choose the entanglement available before Alice makes her particle A interact with the ancilla C. Assuming that the ancilla is initially uncorrelated from particles AB (an assumption that holds in all our examples and that is typically verified experimentally) we find that $E_{AC:B} \leqslant E_i$. The final entanglement is the one available after C reaches Bob's laboratory and he applies a general local transformation on his particles in order to localize the entanglement established between the laboratories into the state of the principal system, i.e., $E_f \leqslant E_{A:CB}$. We therefore conclude that

$$E_f - E_i \leqslant E_{A:CB} - E_{AC:B} = E_{\text{fin}} - E_{\text{in}} = \Delta E. \tag{2}$$

In our nomenclature E_f and E_i indicate, respectively, the final and initial entanglement between the principal subsystems, A and B, whereas E_{fin} and E_{in} denote the final and the initial entanglement between the laboratories, respectively. Nonexcessive protocols for entanglement between the laboratories are also nonexcessive for the principal system, but the excessive protocols for laboratories do not guarantee that the principal system gains entanglement above the communicated one. This is because not all entanglement can be localized into the principal system and some of its initial entanglement might be destroyed by interactions with the ancilla.

III. IDEAL CONDITIONS

In this section we investigate entanglement gain for the ideal case where there is no noise in the communication channel between laboratories. We first study the indirect protocol in Fig. 2, where the state of the whole ABC system is pure. It turns out that the excessiveness depends on the particular entanglement measure being used. We then present a single parameter family of five-qubit states that are shown to provide a single platform exhibiting various possibilities of entanglement gain.

A. Entanglement measures

For subadditive measures proportional to the entropy of subsystems, like the von Neumann entropy and the linear entropy for pure states, we find that the protocols are always nonexcessive. For other measures, such as negativity [21] and logarithmic negativity [21,22], we show that even pure states of sufficiently high dimension give rise to excessive gain.

1. Subadditive measures

We begin by noting that for pure states the condition for nonexcessiveness is equivalent to the subadditivity of measures that characterize pure-state entanglement in terms of the properties of a subsystem. The nonexcessiveness condition reads

$$E_{A:CB} \leqslant E_{AC:B} + E_{AB:C}. \tag{3}$$

If the entanglement involved in the equation above embodies a property of a subsystem, such as $E_{i:jk} = S_{jk}$, with S_{jk} being a property of subsystem jk (here $i,j,k = A,B,C$), we can rewrite this as

$$S_{BC} \leqslant S_B + S_C. \tag{4}$$

This is exactly the subadditivity property, which appears as a necessary and sufficient consequence of the nonexcessive nature of a protocol.

2. *Negativity*

We move to computable entanglement as characterized by negativity [21] and show that all protocols in which entanglement is measured by this quantity are nonexcessive if the dimension of A is less than 3. We provide a simple explicit example of a state in the $3 \times 2 \times 2$ dimension which allows for excessive entanglement gain.

We first prove a lemma that reveals dimensionality-dependent inequality for the negativity, which we then exploit. Recall that the negativity of a bipartite state ρ_{XY} is defined as

$$N_{X:Y} = \frac{\|\rho_{XY}^{PT}\| - 1}{2}, \tag{5}$$

where PT indicates the partial transposition with respect to subsystem X and $\|\sigma\| = \mathrm{Tr}\sqrt{\sigma^\dagger \sigma}$ denotes the trace norm of a generic operator σ.

Lemma 1. The following negativity inequality holds for an arbitrary tripartite system in a pure state,

$$\sqrt{\frac{2}{d_A(d_A - 1)}} N_{A:CB} \leqslant N_{AC:B} + N_{AB:C}, \tag{6}$$

where d_A is the rank of the reduced state of Alice.

Proof. Let us write the global pure state $|\psi\rangle$ in its Schmidt forms for various bipartitions:

$$\begin{aligned} |\psi\rangle &= \sum_{\alpha=1}^{d_A} \sqrt{p_\alpha} |\alpha\rangle_A |\phi_\alpha\rangle_{BC} \\ &= \sum_{\beta=1}^{d_B} \sqrt{q_\beta} |\beta\rangle_B |\chi_\beta\rangle_{AC} \\ &= \sum_{\gamma=1}^{d_C} \sqrt{r_\gamma} |\gamma\rangle_C |\xi_\gamma\rangle_{AB}, \end{aligned} \tag{7}$$

where we have introduced suitable orthonormal bases. In this notation, the respective negativities are

$$\begin{aligned} N_{A:CB} &= \frac{1}{2} \sum_{\alpha \neq a} \sqrt{p_\alpha p_a}, \\ N_{AC:B} &= \frac{1}{2} \sum_{\beta \neq b} \sqrt{q_\beta q_b}, \\ N_{AB:C} &= \frac{1}{2} \sum_{\gamma \neq c} \sqrt{r_\gamma r_c}. \end{aligned} \tag{8}$$

Our starting point is the subadditivity of linear entropy [23], which in present notation reads

$$\sum_{\alpha \neq a} p_\alpha p_a \leqslant \sum_{\beta \neq b} q_\beta q_b + \sum_{\gamma \neq c} r_\gamma r_c. \tag{9}$$

We obtain the lower bound on the left-hand side by noting that the sum can be interpreted as the length of vector $(\sqrt{p_1 p_2}, \sqrt{p_1 p_3}, \ldots, \sqrt{p_{d_A - 1} p_{d_A}})$, whereas its inner product with vector $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ gives the negativity $N_{A:CB}$. By the Cauchy-Schwatrz inequality the lower bound is

$$\frac{4}{d_A(d_A - 1)} N_{A:CB}^2 \leqslant \sum_{\alpha \neq a} p_\alpha p_a. \tag{10}$$

For the upper bound to the right-hand side of (9) consider

$$\begin{aligned} (N_{AC:B} + N_{AB:C})^2 &\geqslant N_{AC:B}^2 + N_{AB:C}^2 \\ &= \frac{1}{4} \sum_{\beta \neq b} \sqrt{q_\beta q_b} \sum_{\beta' \neq b'} \sqrt{q_{\beta'} q_{b'}} \\ &\quad + \frac{1}{4} \sum_{\gamma \neq c} \sqrt{r_\gamma r_c} \sum_{\gamma' \neq c'} \sqrt{r_{\gamma'} r_{c'}} \\ &\geqslant \frac{1}{2} \sum_{\beta \neq b} q_\beta q_b + \frac{1}{2} \sum_{\gamma \neq c} r_\gamma r_c. \end{aligned} \tag{11}$$

The last inequality holds due to the fact that, in the above sums, the combination of two pairs of equal indexes occurs twice, e.g., we get $q_\beta q_b$ by multiplying terms with $\beta = \beta'$ and $b = b'$ and also $\beta = b'$ and $b = \beta'$. All the remaining terms are positive, hence the inequality. By combining the lower bound and the upper bound we arrive at inequality (6).

We are now ready to make our main statement about excessiveness in terms of negativity.

Theorem 1. The following inequality holds for all pure states if and only if subsystem A is a qubit:

$$N_{A:CB} \leqslant N_{AC:B} + N_{AB:C}. \tag{12}$$

Proof. Using (6) with $d_A = 2$ we find exactly the nonexcessiveness condition. If $d_A > 2$, we provide an explicit *minimal* example (in terms of the size of the subsystems) of a state that leads to an excessive protocol. Choose $d_A = 3$, $d_B = d_C = 2$, and consider the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|200\rangle + |001\rangle + |110\rangle), \tag{13}$$

for which the communicated negativity is given by $N_{AB:C} = \sqrt{2}/3 \approx 0.471$ while the negativity gain is $N_{A:CB} - N_{AC:B} = 1 - \sqrt{2}/3 \approx 0.529$.

Theorem 1 has a consequence for the ongoing effort aimed at unifying the current approaches to quantum correlations [24–26]. Their goal in this respect is to quantify general quantum correlations (including entanglement, quantum discord, etc.) with the same mathematical forms, thus allowing for direct comparison of their respective values. We argue that a discord-like quantity on an equal footing with the negativity will not satisfy physically plausible properties. Namely, there are two properties that we would expect from a unified approach: (i) Since all nonclassical correlations of pure states should be due to quantum entanglement, in a unified approach the discordlike quantity should reduce to negativity for pure states; and (ii) we would expect the discord-like quantity to measure the nonclassicality of communication as is the case for other measures [14,15], and therefore in a unified approach the discord-like quantity should bound the negativity gain. However, the violation of inequality (12) shown by higher-dimensional systems implies that there cannot be a discord-like quantity that reduces to negativity for pure states and also respects condition (ii).

3. Logarithmic negativity

A measure related to the negativity is the *logarithmic negativity*, defined as [21,22]

$$L_{X:Y} = \log_2 \|\rho_{XY}^{PT}\| = \log_2(2N_{X:Y} + 1). \quad (14)$$

Similarly to what has been discussed above, we can identify excessive protocols based on the use of the logarithmic negativity and a system A of sufficiently high dimensionality.

Theorem 2. For pure states of three subsystems, A, B, and C, with $d_A = 2$ we have

$$L_{A:CB} - L_{AC:B} \leqslant L_{AB:C}. \quad (15)$$

Proof. The proof follows trivially from inequality (12). Multiply inequality (12) by 2 and add 1 to both sides. Taking the logarithm gives us

$$\log_2(2N_{A:CB} + 1) \leqslant \log_2(2N_{AC:B} + 2N_{AB:C} + 1). \quad (16)$$

The thesis follows if we combine $\log(2N_{AC:B} + 2N_{AB:C} + 1) \leqslant \log(2N_{AC:B} + 1) + \log(2N_{AB:C} + 1)$ with inequality (16).

An extensive numerical analysis performed by testing uniformly picked random pure states suggests that, differently from what has been found for the negativity inequality, (12), inequality (15) always holds for $d_A = 3$. The first example of a pure state that does not satisfy inequality (15) has been encountered for $d_A = 4$ with B and C both being qubits. For instance, for state

$$|\psi\rangle = \frac{1}{\sqrt{103}}(10|000\rangle + |110\rangle + |201\rangle + |311\rangle), \quad (17)$$

the communicated logarithmic negativity is $L_{AB:C} \approx 0.352$, while the corresponding gain in logarithmic negativity is ≈ 0.363, which is excessive. Similarly to violations in the case of negativity, this happens due to the dimension of A. We conjecture that inequality (15) holds in Hilbert spaces of arbitrary dimension where subsystem A has dimension less than 4.

B. Excessive protocols

We now move to a single-parameter family of states which allows for various possibilities of entanglement gain. We emphasize excessive protocols, as they are our main focus here. Recall that a protocol is said to be excessive if $\Delta E > E_{\text{com}}$, where $E_{\text{com}} = E_{C:AB}$ is the entanglement of the carrier with the rest and $\Delta E = E_{\text{fin}} - E_{\text{in}}$, with $E_{\text{in}} = E_{AC:B}$ and $E_{\text{fin}} = E_{A:BC}$, respectively, being the initial and final entanglement between Alice's and Bob's laboratories.

Note that an excessive protocol can, in principle, be realized for all four possible scenarios that correspond to whether the initial and/or communicated entanglement are vanishing. The first example of an excessive protocol demonstrating the possibility to distribute entanglement via a separable carrier state had $E_{\text{in}} = 0$ and $E_{\text{com}} = 0$ [5], whereas in Ref. [15] an example of an excessive protocol was given with $E_{\text{in}} > 0$ and $E_{\text{com}} = 0$. These results may also be understood as a direct consequence of the fact that a tripartite density matrix ρ_{ABC} can have different entanglement across different bipartitions. For example, in the entanglement distribution via the separable states scenario one has $E_{AC:B} = E_{C:AB} = 0$, yet $E_{A:BC} > 0$. In both the above examples the communicated entanglement is 0, as the carrier C remains unentangled, at all times, with other subsystems. We provide here for the first time examples of excessive protocols with nonzero communicated entanglement and give new examples for the scenario where $E_{\text{in}} > 0$ and $E_{\text{com}} = 0$. It should be noted that all such scenarios can be encompassed by a single-parameter family of density matrices, as shown below.

We also observe that the excessive nature of a protocol may depend on whether an intermediate particle (which does not influence the initial entanglement) is sent in advance of the designated carrier. We call this strategy *catalysis of excessiveness*. It can be described as follows. Suppose a protocol P is excessive over a certain range Δ of a given parameter associated with the state (it is not important to specify which parameter). Let the carrier be denoted C. Now consider another protocol, P', where an intermediate particle C' is transmitted without changing the initial entanglement. This implies that the initial entanglement E_{in} is the same for both protocols before the transmission of the designated carrier C. Let Δ' be the range over which P' is excessive. We find not only that P' is excessive over a wider range, that is, $\Delta' > \Delta$, but also that, within the range Δ where P and P' are both excessive, the entanglement gain in P' is greater than the corresponding gain in P. Thus an intermediate carrier, which does not change the initial entanglement, can make an excessive protocol better. We give explicit examples demonstrating this effect later.

Consider a five-qubit density matrix obtained by applying local quantum channels on a five-qubit *absolute maximally entangled* (AME) state [27]. AME states have the property to be maximally entangled across every bipartition. Thus we might expect such states to be more robust to local noise and conceivably good candidates to exhibit excessive protocols. The five-qubit pure AME state is given by

$$\begin{aligned}|\psi\rangle = \tfrac{1}{4}(&|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\ &+ |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ &+ |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\ &+ |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle). \quad (18)\end{aligned}$$

We then construct the density matrix $\rho(q)$ resulting from the application of two specific local quantum channels to the first two qubits of such state. Such channels, which we label $\Lambda_{1,2}(k = 1,2)$, are defined in terms of their respective Kraus operators as

$$K_0^{(1)} = \frac{1}{\sqrt{2}}\mathbb{1}, \quad K_i^{(1)} = \frac{1}{\sqrt{6}}\sigma_i,$$
$$K_0^{(2)} = \sqrt{q}\,\mathbb{1}, \quad K_i^{(2)} = \sqrt{\frac{1-q}{3}}\,\sigma_i$$

for $q \in [0,1]$ and $i = x,y,z$. Correspondingly, the five-qubit state is now

$$\rho(q) = (\Lambda_1 \otimes \Lambda_2 \otimes \mathcal{I}_3 \otimes \mathcal{I}_4 \otimes \mathcal{I}_5)[|\psi\rangle\langle\psi|]. \quad (19)$$

Channel Λ_1 is always EB, and thus the first qubit becomes unentangled with the rest of the system. Channel Λ_2 is EB for $0 \leqslant q \leqslant 0.5$. The application of such channels breaks the

TABLE I. Separability properties of the five-qubit state $\rho(q)$ defined in Eq. (19). The first column specifies the relevant bipartitions of the five qubits, whereas the other columns indicate whether, in the corresponding bipartition, the state is entangled. For the top two bipartitions the separability changes as the parameter q is tuned above 0.5.

Partition	$0 \leqslant q \leqslant 0.5$	$0.5 < q \leqslant 1$
12:345	Separable	Entangled
2:1345	Separable	Entangled
1:2345	Separable	Separable
3:1245	Entangled	Entangled
13:245	Entangled	Entangled
123:45	Entangled	Entangled

symmetry initially present in the AME state across its bipartitions. The excessive nature of the entanglement distribution protocols that we describe is then the consequence of this breakdown of symmetry. Table I summarizes the separability properties of the state across the relevant bipartitions.

Having set the resource state to use, we now present new excessive protocols for various combinations of initial and communicated entanglement. In each case we give two examples to highlight the fact that, given a system of many particles in some specified quantum state, there can be different tripartite configurations giving rise to an excessive protocol of the *same kind*.

1. Excessive protocols with $E_{\mathrm{in}} = 0$ and $E_{\mathrm{com}} > 0$ ($E_{\mathrm{in}} > 0$ and $E_{\mathrm{com}} = 0$)

Let us group the five qubits into the subsystems $A = \{2,4,5\}$, $B = \{1\}$, and $C = \{3\}$. Alice initially holds subsystems A and C, whereas Bob holds B. As reported in Table I, there is no entanglement between the laboratories in this configuration, i.e., $L_{AC:B} = 0 \forall q$. However, Fig. 3 demonstrates that sending C through a noiseless channel generates more final entanglement than what was communicated (in terms of logarithmic negativity). More specifically,

$$L_{A:BC} - L_{AC:B} > L_{AB:C} \quad \text{for} \quad q \neq \left\{0,\tfrac{1}{4},1\right\}. \tag{20}$$

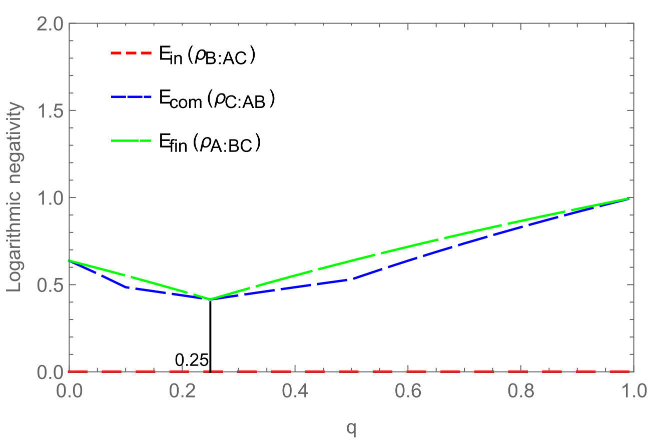

FIG. 3. Excessive protocols with $E_{\mathrm{in}} = 0$ and $E_{\mathrm{com}} > 0$ realized through a local channel-affected five-qubit AME state. For all values of q except $q = \{0,1/4,1\}$ we see that $E_{\mathrm{fin}} > E_{\mathrm{com}}$. Partitions used here are $A = \{2,4,5\}$, $B = \{1\}$, and $C = \{3\}$. By swapping qubits 1 and 3 we obtain excessive protocols with $E_{\mathrm{in}} > 0$ and $E_{\mathrm{com}} = 0$.

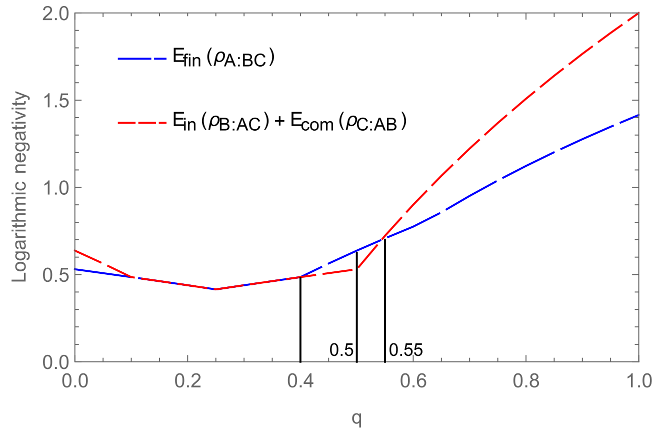

FIG. 4. Excessive protocols with nonvanishing communicated entanglement. For $0.4 < q \leqslant 0.5$ the protocol is excessive with $E_{\mathrm{in}} = 0$ and $E_{\mathrm{com}} > 0$. For $0.5 < q < 0.55$ the protocol is excessive while satisfying $E_{\mathrm{in}} > 0$ and $E_{\mathrm{com}} > 0$. Partitions here are $A = \{1,4,5\}$, $B = \{2\}$, and $C = \{3\}$.

The same kind of excessive protocol is obtained for yet another grouping of qubits: $A = \{1,4,5\}$, $B = \{2\}$, and $C = \{3\}$. As reported in Table I, the initial entanglement vanishes for $0 \leqslant q \leqslant 0.5$, whereas the carrier particle is entangled for all q. Figure 4 reveals that this protocol is thus excessive for $0.4 < q \leqslant 0.5$.

A new family of examples with $E_{\mathrm{in}} > 0$ and $E_{\mathrm{com}} = 0$ is obtained from the cases studied above by simply exchanging the roles of subsystems B and C. Protocols that were excessive before are still excessive after the swap, as can be verified by rewriting inequality (20) as $L_{A:BC} - L_{AB:C} > L_{AC:B}$. By our analysis above, the swap $B \leftrightarrow C$ also exchanges E_{in} and E_{com}.

2. Excessive protocols with $E_{\mathrm{in}} > 0$ and $E_{\mathrm{com}} > 0$

Consider again the grouping $A = \{1,4,5\}$, $B = \{2\}$, and $C = \{3\}$ of the particles in the state in Eq. (19). This time, we take into account the range $0.5 < q \leqslant 1$, in which the laboratories are initially entangled, i.e., $L_{AC:B} > 0$. Under these conditions the carrier particle is also entangled, i.e., $L_{AB:C} > 0$. Figure 4 reveals that the protocol is excessive for $0.5 < q < 0.55$. Similar conclusions hold under the swap $B \leftrightarrow C$.

3. Catalysis of excessiveness

A form of catalysis of entanglement is implicitly present in the protocol for distribution with separable states [5]. There, first, particle B, which is separable initially from A and C, is sent to Bob's laboratory keeping vanishing entanglement between Alice and Bob. Next, Alice sends C, while $E_{AB:C} = 0$. However, $E_{A:CB} > 0$ in the end. Particle B worked as a *catalyst*.

Despite the analogies, the phenomenon dubbed here catalysis of excessiveness is more general. In order to illustrate this, we discuss another example starting from the state $\rho(q)$ as in Eq. (19). Let us now group the five qubits as $A = \{4,5\}$,

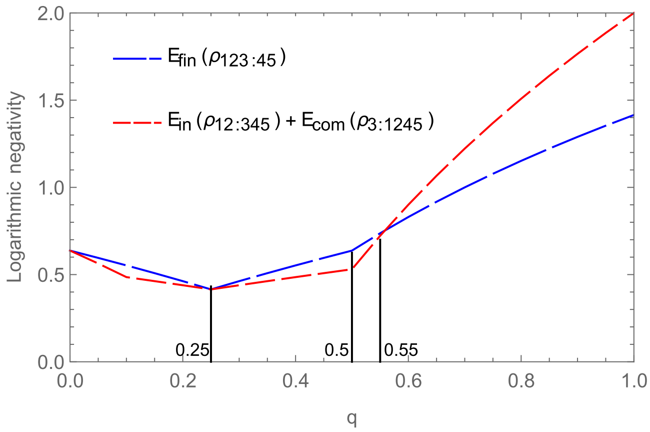

FIG. 5. Catalysis of excessiveness. Comparison of this plot with Fig. 4 shows that sending qubit 1 in advance increases entanglement gain between the laboratories and the range of q over which the protocol is excessive while having no influence on the initial entanglement.

$B = \{1,2\}$, and $C = \{3\}$. This implies that, compared with the example in the previous subsection, qubit 1 has been sent by Alice to Bob *before* the protocol begins. According to Table I, the initial states in both cases (before and after sending qubit 1) have the same separability properties and one can verify that the actual logarithmic negativities are exactly the same. Therefore, the communication of qubit 1 has no influence on the initial entanglement between the laboratories.

However, if we now send subsystem C to Bob, we notice a different behavior of the entanglement gain. By computing the amount of initial, communicated, and final entanglement, we find the results displayed in Fig. 5, which show that the predelivery of a qubit to Bob's laboratory does influence the performance of entanglement distribution. Indeed, by comparing Figs. 4 and 5, we see that while in the former the protocol is excessive only for $0.4 < q < 0.55$, the latter reveals excessiveness for all $q < 0.55$ (except $q = \{0,0.25\}$). Since q parametrizes local noise acting on an individual qubit of the register, this can be regarded as an increase in the robustness of the protocol. Furthermore, the actual gain of entanglement in the excessive part of the protocol is larger if qubit 1 is communicated in advance, thus clarifying the catalytic role of this prestep in the protocol.

IV. NOISY ENVIRONMENTS

One of the main obstacles to the generation and preservation of entanglement is the presence of noisy environments. As full noise avoidance appears to be too demanding or costly to embody a viable way to circumvent the problem, a potential approach to facing the challenge of noisy entanglement distribution is to design protocols able to work well under such nonideal conditions.

The noise affecting the communication channel that connects two laboratories will interfere mainly with the entanglement communicated between them. This suggests that excessive protocols, which have a small amount of communicated entanglement compared to the gain, could work better than nonexcessive ones. In this section we verify the efficiency of concrete indirect protocols for entanglement distribution in the presence of three typical quantum noises. We also address the most extreme case of noisy channels, i.e., EB channels. The results that have been achieved through our analysis suggest that excessive protocols often allow for significant amounts of distributed entanglement, even under the presence of rather strong noise. As a quantitative measure of entanglement we use negativity and we focus on the situation where the three subsystems involved in the distribution schemes are all qubits. According to the analysis in Sec. III, the communicated entanglement always exceeds the entanglement increment if the three-qubit system is in a pure state, but since noise will necessarily mix the system, excessive distribution becomes possible.

A. Noisy channels

We begin by introducing three standard channels modeling environmental noise acting on two-level systems: the dephasing channel, depolarizing channel, and amplitude damping channel. For a unified description, all of them are represented in terms of Kraus operators involving a single parameter that characterizes the strength of the noise.

1. *Entanglement-breaking channel*

In general, if a channel produces separable output independent of the input state, it is said to be entanglement breaking. As proven in Ref. [28], a channel is EB if and only if its action on any state can be written as a measure-and-prepare positive operator-valued measurement (POVM) on the particle that goes through the channel. In the tripartite scenario at the core of our work, the channel Φ_C acts on the ancillary particle C that is communicated between the laboratories, so that the resulting state can be written as

$$(\mathcal{I}_{AB} \otimes \Phi_C)(\rho_{ABC}) = \sum_n p_n \rho_{AB|n} \otimes \gamma_n, \tag{21}$$

where p_n are the probabilities associated with the measurement outcomes that are part of the POVM performed on C, and $\rho_{AB|n}$ are the states of AB conditioned to the outcomes of the POVM measurement. Finally, γ_n are rank 1 projectors (pure states) that one prepares on C, depending on the result of the measurement.

Each of the channels introduced in the following subsections has a critical value of its characteristic noise parameter above which it becomes EB.

2. *Dephasing channel*

The dephasing channel captures the loss of coherence in a preferred basis. The strength of this loss is given by the parameter $\delta_{\rm ph}$, and if the preferred basis is chosen to be that embodied by the eigenbasis of the σ_z Pauli matrix, the Kraus operators of the dephasing channel take the form

$$K_0^{(\rm ph)} = \sqrt{1 - \frac{\delta_{\rm ph}}{2}}\,\mathbb{1}, \quad K_1^{(\rm ph)} = \sqrt{\frac{\delta_{\rm ph}}{2}}\,\sigma_z. \tag{22}$$

This channel is EB when $\delta_{\rm ph} = 1$.

3. *Depolarizing channel*

The depolarizing channel describes the loss of coherence in any basis. It is defined by the Kraus operators

$$K_0^{(\mathrm{pol})} = \sqrt{1-\delta_{\mathrm{pol}}}\,\mathbb{1}, \quad K_{x,y,z}^{(\mathrm{pol})} = \sqrt{\frac{\delta_{\mathrm{pol}}}{3}}\sigma_{x,y,z}. \tag{23}$$

The channel is EB for $\delta_{\mathrm{pol}} \in [1/2,1]$.

4. *Amplitude-damping channel*

The amplitude-damping channel describes energy dissipation from the system. The Kraus operators for this channel are

$$\begin{aligned} K_1^{(\mathrm{ad})} &= |0\rangle\langle 0| + \sqrt{1-\delta_{\mathrm{ad}}}|1\rangle\langle 1|, \\ K_2^{(\mathrm{ad})} &= \sqrt{\delta_{\mathrm{ad}}}|0\rangle\langle 1|. \end{aligned} \tag{24}$$

Similarly to the dephasing channel, the amplitude-damping channel is EB for $\delta_{\mathrm{ad}} = 1$.

B. Indirect distribution via noisy channels

In the numerical studies presented in this subsection we extend the examples presented in Ref. [11]. Consider the scenario depicted in Fig. 6. Particles A and B managed by Alice and Bob are already at their respective laboratories. We assume that they are prepared in the Werner state

$$\alpha_{AB} = p|\phi_+\rangle\langle\phi_+| + \frac{(1-p)}{4}\mathbb{1}_4, \tag{25}$$

where $|\phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a Bell state ($|0\rangle$ and $|1\rangle$ are the eigenstates of the local σ_z operator) and $\mathbb{1}_4$ is the identity matrix for a two-qubit system. For $p \leqslant 1/3$, the Werner state is separable. Particle C starts in a state of the form [9]

$$\alpha_C = \tfrac{1}{2}(\mathbb{1}_2 + s\sigma_x), \tag{26}$$

where $s \in [0,1]$. We choose the interaction between A and C to be the controlled-phase (c-phase) gate, as in Ref. [9]. Extensive numerical study suggests that this is the optimal choice. The noisy mechanism is supposed to affect the channel that connects the remote laboratories and, thus, acts on C as it travels from Alice to Bob. The c-phase interaction is assumed to take place at Alice's laboratory. We then calculate the entanglement gain of this indirect protocol.

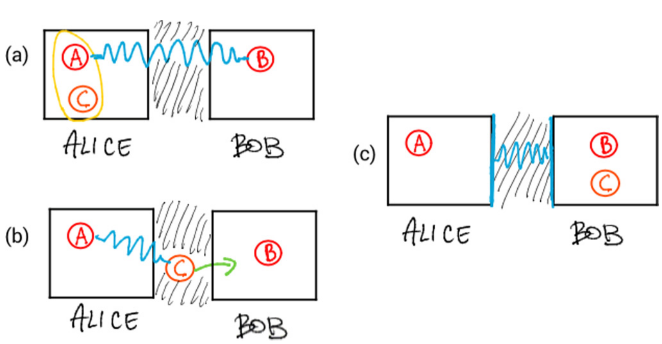

FIG. 6. Indirect entanglement distribution via the noisy channel. (a) A and B are initially already displaced and correlated when C interacts locally with A. (b) C is communicated via the noisy channel to Bob's laboratory. (c) The final entanglement between the laboratories is $E_{A:BC}$.

The results of our quantitative analysis are presented in Fig. 7. Figures 7(a)–7(c) demonstrate that the largest entanglement gain is achieved when particle C is prepared in a pure state, independently of the type of noise in the channel.

Furthermore, this maximal gain is achieved via a nonexcessive protocol, again regardless of the applied noise. However, the parameter regions where the protocol is nonexcessive are very small. If the noise is not too weak, and if Alice is not able to put the ancilla C in a very pure state, then there will be more entanglement gain than communicated. Note that this is the case for almost all values of the noise parameters. Complementary results are obtained if the negativity gain is calculated as a function of the entanglement admixture in the Werner state, as shown in Figs. 7(d)–7(f). This time the parameter range of excessive distribution is reduced, although it always contains the protocols that achieve the maximum gain. Note that, independent of the noisy channel and the strength of the noise, the largest negativity increment is obtained for initial Werner states lying on the separability border, i.e., for $p = 1/3$. To some extent, this can be intuitively explained: the degree of entanglement in a maximally entangled state cannot be improved and the increment of entanglement of a "deeply" separable state first has to overcome the distance to the separability border.

C. Direct-then-indirect distribution via noisy channels

The protocol described in the previous subsection assumes that particles A and B are already far from each other, yet prepared in a (partially) entangled state. Clearly, this has a cost, as Alice and Bob should be able to generate it using a noisy channel that cannot be bypassed in our scenario. In order to include the cost of preparing such initial state, we now study the protocol illustrated in Fig. 8. Alice, who initially holds both particle A and particle B, prepares the Werner state in Eq. (25). Next, particle B is communicated via the noisy channel to Bob. In addition to this direct protocol, Alice and Bob run the indirect protocol using the initial state of the ancilla given in Eq. (26) and the c-phase gate, as illustrated before.

In Fig. 9 we present the negativity gain in the indirect protocol alone, i.e., the entanglement achieved after the direct distribution step is subtracted from the final entanglement between the laboratories. This allows us to easily compare this situation with the entanglement gains presented in Fig. 7. First we note that in both cases the range of parameters giving excessive gain is very large. This can be quantified by the areas in the plots in Figs. 7 and 9 within the bold black contour and interpreted as high robustness of excessive protocols against noise.

Other similarities to the protocols of the previous subsection relate to the excessiveness of the optimal entanglement gain. As demonstrated in Figs. 9(a)–9(c) the highest gain is obtained starting with pure ancilla states and the protocol is nonexcessive, independent of the type of noise. However, it should be stressed that extra communication of particle B over noisy channel leads to more mixed ancillary states, while the indirect protocol is still nonexcessive. The second similarity is that the best protocols are always excessive

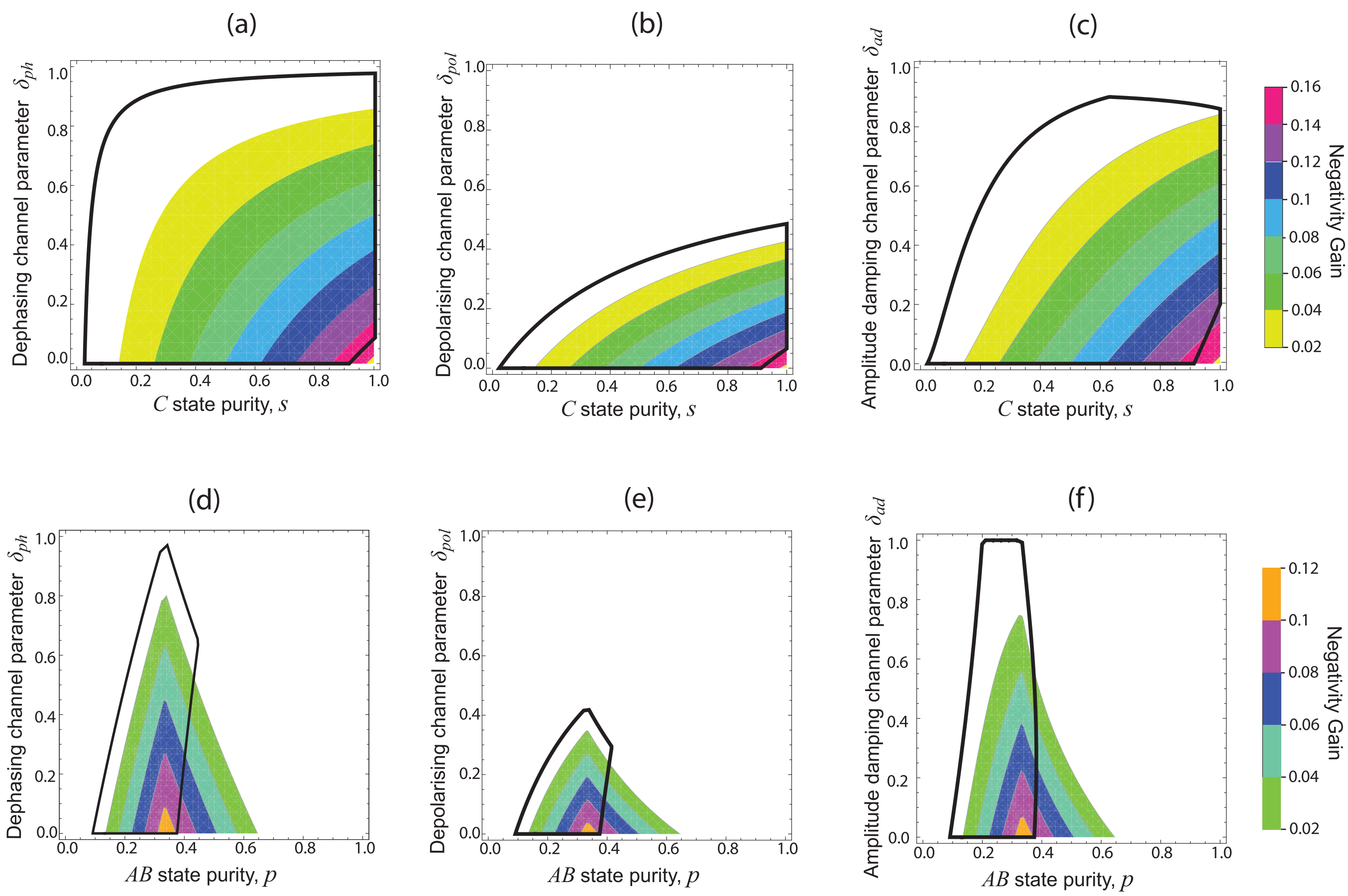

FIG. 7. Entanglement gain in the indirect protocol in Fig. 6. Results for different noises are presented in columns. Thick black lines include regions in which the protocol is excessive. The principal system begins in a Werner state, Eq. (25). (a)–(c) The gain is presented as a function of the purity of the ancilla, s in Eq. (26). For these plots the entanglement admixture in the Werner state is fixed at $p = 0.34$, but essentially the same qualitative results are obtained for other admixtures. The optimal gain is reached for a pure state of C and lies outside the excessive border for low noise parameters. (d)–(f) Entanglement gain is presented as a function of the purity of the principal system, p in Eq. (25). For these plots $s = 2/3$, but again, results are similar for different values of s. Note that for all noises the optimal protocol is for $p = 1/3$ (disentangled initial state) and it is always excessive (whenever effective).

[cf. Figs. 9(d)–9(f)]. Summing up, whether the protocol is excessive or nonexcessive depends on easily controllable parameters, such as the purity of the state of particle C and parameter p in the state of the pair AB.

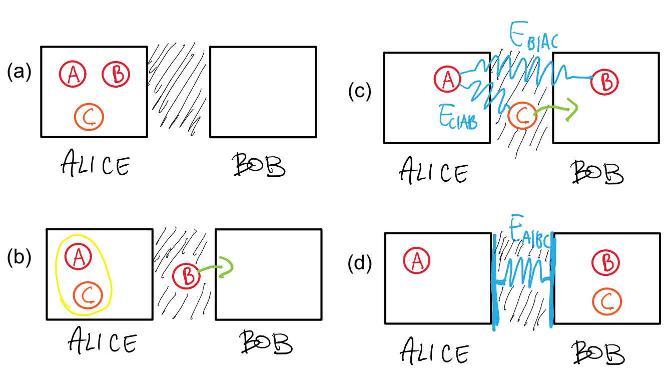

FIG. 8. Direct-then-indirect distribution via the noisy channel. (a) Now A and B are initially correlated locally in Alice's laboratory. (b) While B travels through the channel (direct protocol), A and C interact with each other. (c) Finally, C reaches Bob, (d) changing the entanglement between the laboratories.

Not all the features of the present protocol are the same as in the one illustrated in the previous subsection. The maximum gain of negativity now depends on both the purity of the Werner-state resource and the actual amount of noise in the channel. For perfect channels, it is always best to begin with the Werner state at the border of separability. For noisy channels, the entanglement admixture in the initial Werner state corresponding to the largest entanglement gain depends on the strength of the channel. For stronger noise, i.e., larger δ_{ch}, one should begin with larger entanglement in the Werner state as part of it would be lost during the initial transmission of B. However, as intuition suggests, such initial entanglement cannot be too large, as it is very difficult to improve the degree of entanglement in states that are already highly entangled.

Finally, we note that, starting with the direct distribution of mixed Werner states, there is a range of noise parameters leading to no direct entanglement gain, although the channels are not EB (see Sec. IV E). This is clear for initially separable Werner states but also happens if they are weakly entangled. If Alice can only prepare such a Werner state, the indirect protocols provide the only means of entanglement creation between the laboratories, and for certain noise and state parameters, the excessive distribution is the only viable option.

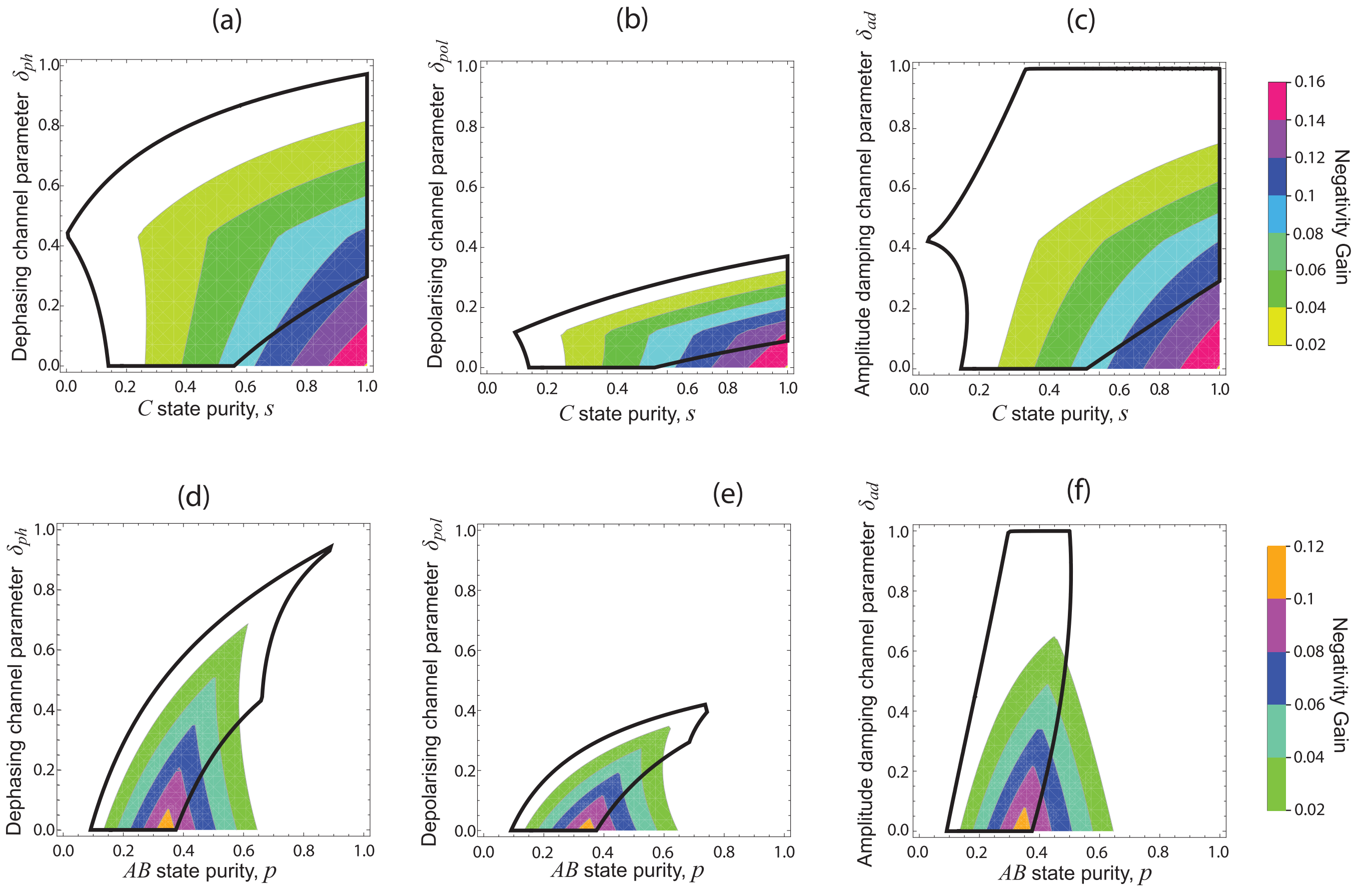

FIG. 9. Entanglement gain achieved solely in the indirect protocol in Fig. 8, i.e., the final negativity minus the negativity between the laboratories after the direct protocol. The region delimited by the thickblack line shows where the protocol is excessive. Results for different noises are presented in columns. (a)–(c) The parameter of the initial Werner state is $p = 0.34$, but the same qualitative results are obtained for different values of p. Note that, similarly to Fig. 7, the optimal gain is achieved with C in a pure state. (d)–(f) The negativity gain is presented as a function of the parameter of the Werner state of A and B, i.e., p in Eq. (25). Ancilla C is initially in state (26) with $s = 2/3$. Again, qualitatively the same results are seen for different values of s. In this case, the largest gain is achieved by excessive protocols (whenever effective), but differently from Fig. 7 the optimal value of p now depends on the strength of the noise.

D. Direct-then-indirect distribution with noisy laboratories and via noisy channels

Our last numerical example proceeds towards the consideration of more realistic experimental situations. In addition to a noisy channel connecting them, we now also include local noise affecting the laboratories of Alice and Bob, as Fig. 10 illustrates. The protocol goes as follows. Alice begins with three particles: AB, prepared in the Werner state of Eq. (25), and C, prepared in state $(|0\rangle + |1\rangle)/\sqrt{2}$. The Werner state can be regarded as an output of some nonperfect entangling procedure. Next we assume that A and C interact via a c-phase gate which is instantaneous and ideal. Then particle B travels to Bob via the noisy channel, whereas A and C are independently affected by local noise. Finally, ancilla C experiences the channel noise, particle A experiences the noise in Alice's laboratory, and particle B experiences the noise in Bob's laboratory.

A representative case is when both local noises are amplitude damping, e.g., thermal baths, while the noise in the channel can be different. Figure 11 shows that in the present case the excessive protocols are also very robust as characterized by the range of noise parameters for which there is a gain in negativity. The protocols giving the highest entanglement gain are always found to be excessive. Note also that the gain is one order of magnitude smaller than

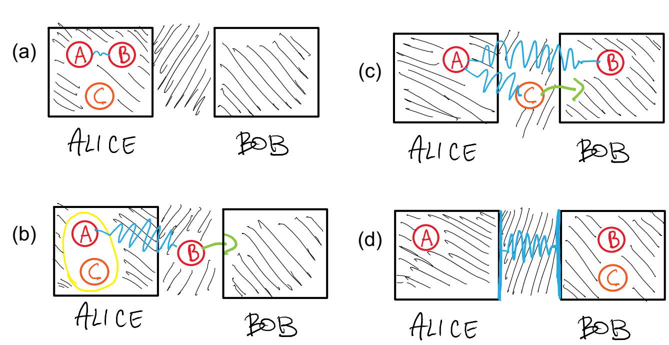

FIG. 10. Entanglement distribution in the presence of local noises as well as a noisy channel. (a) All qubits are initially in Alice's laboratory. (b) B travels to reach Bob, while A and C stay in the noisy laboratory. (c) Finally, C travels through the channel while A and B are affected by their respective local noises. (d) We compare the final entanglement between the laboratories with the entanglement between them after the direct protocol, i.e., (c).

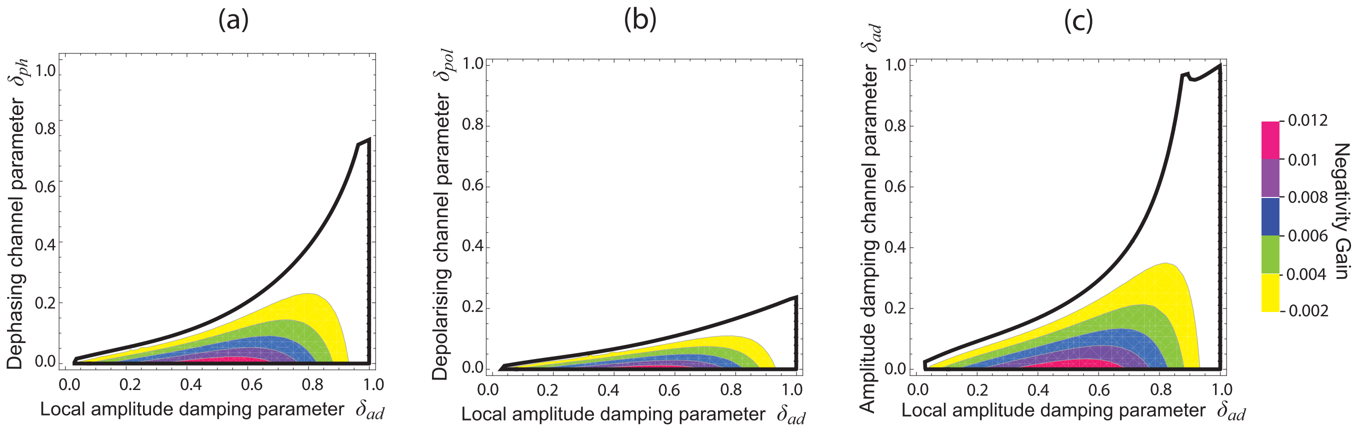

FIG. 11. Entanglement gain achieved in the indirect protocol described in Fig. 10. The parameter for the initial Werner state is $p = 0.34$, while C is prepared in a pure state. The local noises of Alice and Bob are chosen as amplitude damping noises and assumed here to have the same strength, given on the horizontal axis. Along the vertical axes we present the strengths of different channel noises. Within the thick black line the protocol is excessive, i.e., the communicated entanglement is smaller than the gain. There are many pairs of noise parameters that allow excessive protocols to take place. Note that the points of largest gain, for a local amplitude damping parameter $\delta_{\mathrm{ad}} \sim 0.5$, are inside the excessive region. The gain in negativity is, however, much smaller than in the absence of local noise (see Figs. 7 and 9).

in cases without local noise. Altogether, although excessive protocols seem unusual, they are actually quite relevant when entanglement is gained via simple protocols operating under natural noisy conditions.

E. No distribution via entanglement-breaking channels

The most intriguing feature of excessive protocols is the possibility of entanglement gain even if no entanglement is communicated. The existence of such protocols has been known since the work by Cubitt *et al.* [5]. A natural question arises in the context of noisy channels: If, in order to gain entanglement, no entanglement has to cross the channel, could EB channels allow for excessive entanglement gain?

A positive answer to this question would provide a striking example of the usefulness of indirect and excessive protocols. Unfortunately, the answer is negative, as can be seen from Eq. (21). The same effect as the action of EB channels could be obtained by the following local operations and classical communication. Instead of sending C via the channel, Alice performs (in her laboratory) the POVM measurement corresponding to the EB channel. She then sends the (classical) outcome to Bob's location, where an ancillary system D is prepared in the corresponding states γ_n. Alice and Bob do not know what the actual outcome of the measurement is. Therefore, they assign a density operator as in Eq. (21) to particles A, B, and D. As only classical information is transmitted, entanglement does not grow.

A more formal proof emphasizing that entanglement gain is calculated for different bipartitions is given by the following inequality:

$$E'_{A:BD} - E_{AC:B} \leqslant E'_{AC:BD} - E_{AC:BD} \leqslant 0. \tag{27}$$

The inequality on the right means that local operations and classical communication do not increase entanglement, where $E_{AC:BD}$ denotes entanglement before the action of the channel and $E'_{AC:BD}$ that after the channel. The left inequality is obtained from $E_{AC:B} = E_{AC:BD}$, as initially Bob's ancilla is completely uncorrelated, and using $E'_{A:BD} \leqslant E'_{AC:BD}$, due to tracing out one subsystem.

The case of EB channels clearly illustrates that, although communicated quantum discord is necessary for entanglement gain [14,15], it is not a tight bound on entanglement gain in every distribution protocol [9]. The state resulting from EB channels can possess some discord as measured on the ancilla, but this can be thought of as being locally created by a device in Bob's laboratory, which in fact is fed with purely classical communication.

V. CONCLUSIONS

We have classified protocols for entanglement gain into direct and indirect, depending on whether entanglement is generated via mutual interaction or with the help of ancillas, and, further, into excessive and nonexcessive, depending on whether entanglement gain exceeds the amount of communicated entanglement. Analytical examples are provided illustrating the various protocols. This analysis has been complemented by a numerical study showing the usefulness of excessive protocols in the presence of noise.

These results will be of use in quantum information science where distributing entanglement is a prerequisite for many relevant tasks. Achieving this in a cheap and reliable way is essential to the development of future quantum technologies. The results also reinforce the role of quantum discord as a beneficial and practically relevant quantity [29–43]: As discord appears to bound the entanglement gain and to allow for excessive entanglement distribution, it emerges as a key player of fundamental relevance in the context of quantum communication and networking.

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